Intermingled basins and on-off intermittency in a multistate system

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We consider a dynamical system containing infinite low-dimensional symmetric invariant subspaces, each of which has a chaotic state. Intermingled basins are found between these multiple chaotic states when they are stable in the subspaces. As a parameter of the system varies, the largest Lyapunov exponent transverse to the invariant subspace can change from negative to positive; then, the system dynamics changes from an intermingled basin state to a multistate on-off intermittency. The statistical behavior and physical transportation property for different dynamic states are investigated in detail.

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I. INTRODUCTION

Recently, the phenomena of riddled basin and on-off intermittency in chaotic dynamical systems have become an area of intensive study [1–5]. Near certain critical situation, the basin of a chaotic attractor in an invariant subspace may be riddled with holes belonging to the basin of another attractor. The phenomenon of a riddled basin is in fact related to the phenomenon of on-off intermittency [6–20], which refers to the situation where some dynamical variable exhibits its two distinct states with time evolution: one is the ''off'' state, where the variable remains approximately a constant, and the other is the ''on'' state, where the variable temporarily bursts out of the ''off'' state.

A more extreme type of riddled basin structure called intermingled basins has been investigated [18–20]. In this case, two basins of attraction are riddled by each other, which usually occurs when there are two invariant subspaces in the system which are attracting. There is no other attractor apart from the mutually riddled ones. Related to the intermingled basins, a type of intermittent behavior, referred to as two-state on-off intermittency, can be seen. Lai and Grebogi [19] have studied these phenomena, using a mechanical system where a particle moves under the influence of a two-well potential in the (x,y) plane, subjected to friction and periodic forcing of the form $f_0 \sin(wt)$ in the x direction.

In this paper we go further from [19] to consider a system with infinite subspaces, which shows multistate intermingled basins and multistate on-off intermittency. The significant points are that we are able to study the statistical behavior among the infinitely many intermingled basins and reveal the transport property in these distributed invariant subspaces when on-off intermittency sets in, which are of physical importance and beyond observations so far in the study of two- or finite-state intermingled basins and on-off intermittencies.

II. MODEL

We consider a system with a particle moving in the following potential:

$$V(x,y)=(1-x^2)^2+\cos^2 y(x-d)+b\cos^4 y, \tag{1}$$

which is periodic in the y direction. The equation of motion reads

$$\frac{d^2}{dt^2}\mathbf{X}=-\gamma\frac{d}{dt}\mathbf{X}-\nabla V(\mathbf{X})+f_0\sin(wt)\mathbf{X}_0, \tag{2}$$

where γ is the friction coefficient, $\mathbf{X}=(x,y)$, and $\mathbf{X}_0$ is the unit vector in the x direction. We can specify Eq. (2) by five first-order autonomous differential equations in terms of the dynamical variables $x, v_x=dx/dt$, $y, v_y=dy/dt$, and $z=\omega t$:

$$\begin{aligned}\frac{d}{dt}x&=v_x,\\ \frac{d}{dt}v_x&=-\gamma v_x+4x(1-x^2)-\cos^2 y+f_0\sin z,\\ \frac{d}{dt}y&=v_y,\\ \frac{d}{dt}v_y&=-\gamma v_y+\sin(2y)(x-d)+4b\sin y\cos^3 y,\\ \frac{d}{dt}z&=w.\end{aligned} \tag{3}$$

Note that at $y=n\pi-\pi/2$ $(n\in\mathbb{Z})$, we have $\cos y=0$. Therefore, the sets

$$y=n\pi-\pi/2(n\in\mathbb{Z}),\quad v_y=0 \tag{4}$$

are the invariant subspaces of the system, and the number of these invariant subspaces is infinite. The equation of motion in any such subspace describes a forced-damped Duffing oscillator:

$$\begin{aligned}\frac{d}{dt}x&=v_x,\\ \frac{d}{dt}v_x&=-\gamma v_x+4x(1-x^2)+f_0\sin z,\\ \frac{d}{dt}z&=w.\end{aligned} \tag{5}$$

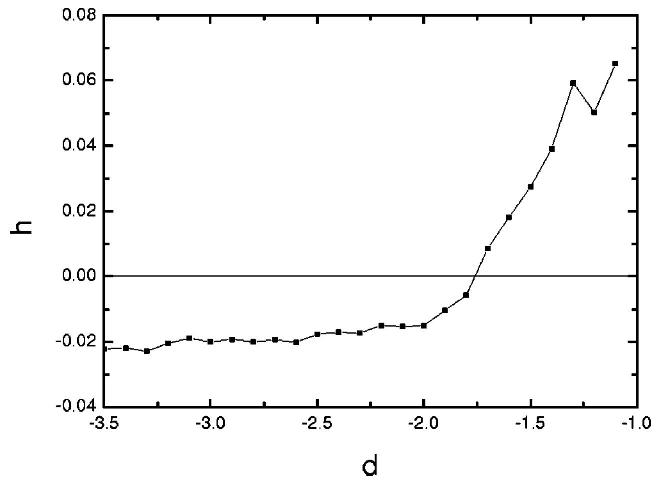

FIG. 1. The largest Lyapunov exponent $h_\perp$ transverse to the invariant subspaces is plotted vs the parameter d for $d \in [-3.5, -1.0]$, in which $h_\perp$ changes from negative to positive at $d_c = -1.75$.

We set $f_0 = 2.3$, $\gamma = 0.05$, and $\omega = 3.5$; then, the attractor in these invariant subspaces is chaotic. To ensure that the full system does not have other attractors, we choose $b = 0.008$. It should be noticed that when $y = n\pi (n \in \mathbb{Z})$, we have $\sin y = 0$; then, the sets $y = n\pi (n \in \mathbb{Z})$, $v_y = 0$ are also invariant subspaces. However, the motions on these subspaces are always unstable in the transverse direction under our parameters, which will not be considered in this paper. In the following we will consider intermingled basins and on-off intermittency in system (3). Since we have an infinite number of invariant subspaces, some new and interesting problems arise, such as probabilities for the system to reach different subspaces when an intermingled basin appears and the transport behavior of the system when on-off intermittency occurs.

III. STATISTICS OF MULTI-INTERMINGLED BASINS

The largest Lyapunov exponent $h_\perp$ transverse to the invariant manifold (4) can be easily computed. We obtain

$$\frac{d}{dt}\delta y = \delta v_y ,$$

$$\frac{d}{dt}\delta v_y = -\gamma \delta v_y - 2[\hat{x}(t) - d]\delta y, \tag{6}$$

where $\hat{x}(t)$ is a chaotic trajectory produced by Eq. (5) in the invariant subspaces, acting like a driving signal in Eq. (6).

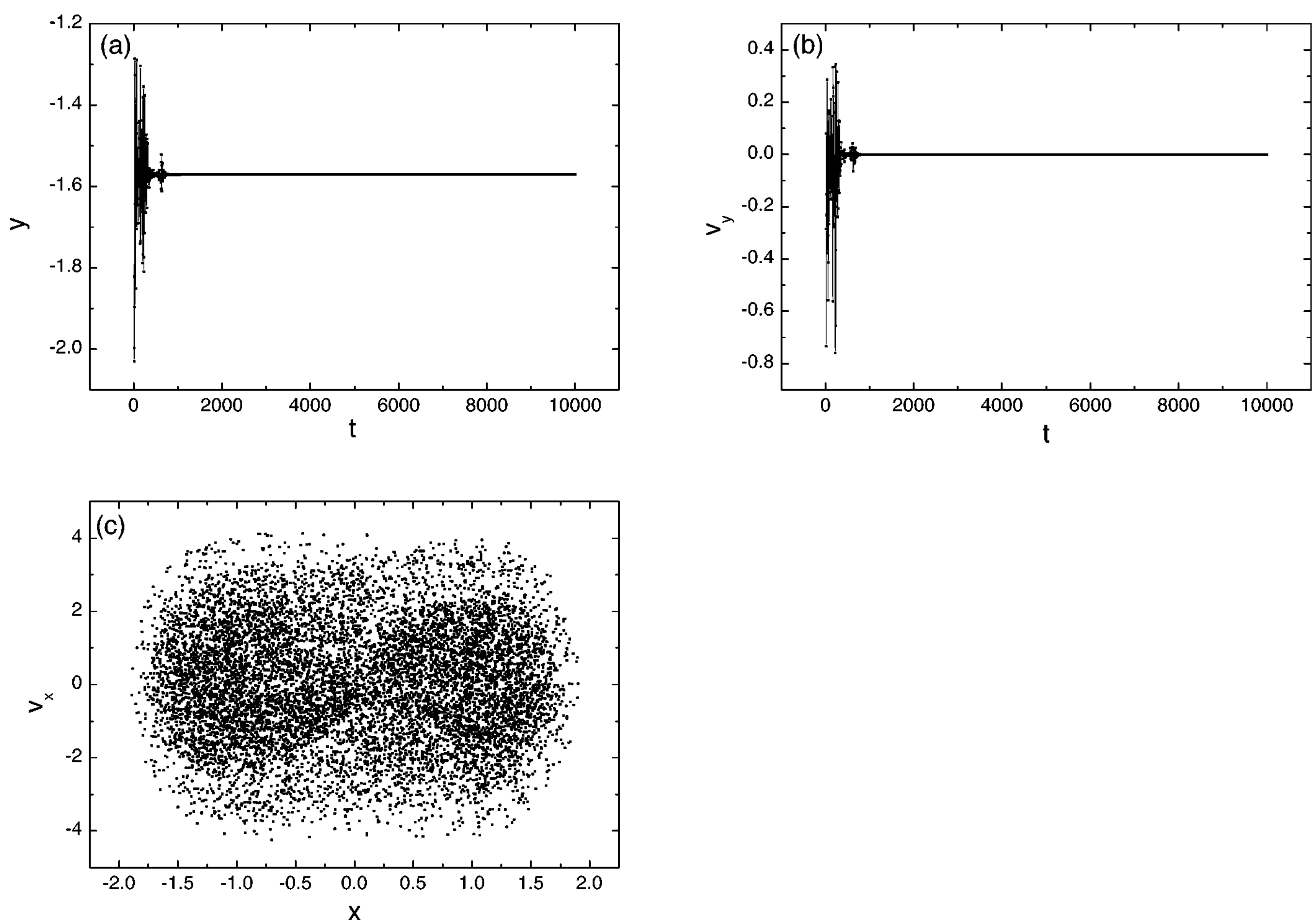

FIG. 2. (a), (b) The typical time sequencies of y and v_y vs t, respectively, with $d = -1.85$. (c) The projection of the system motion in the x-v_x plane.

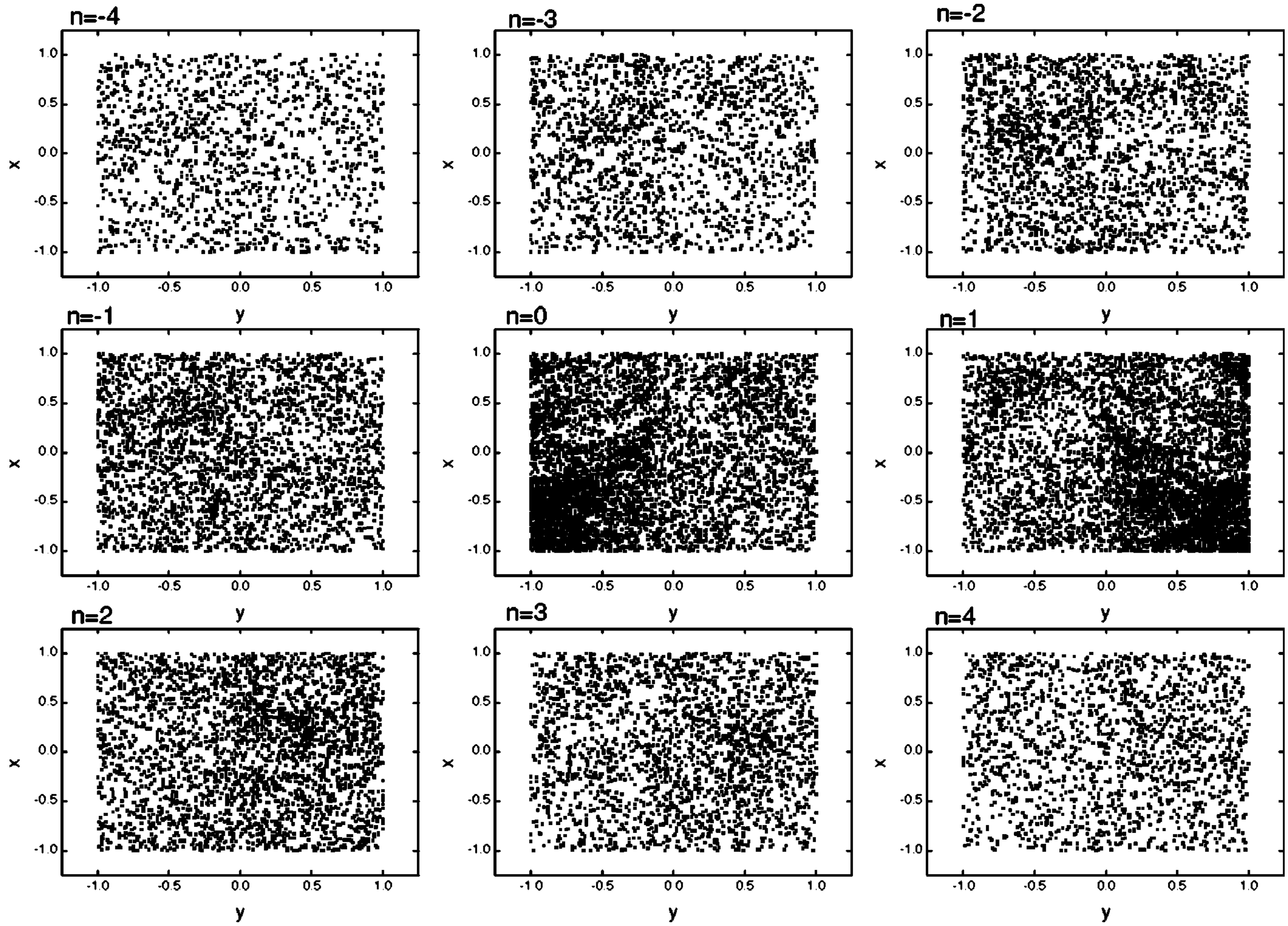

FIG. 3. The basins of attractors at $y=n\pi-\pi/2(n\in\mathbb{Z})$, $v_y=0$ plotted for different state number n (n is from -4 to 4). For the indicated initial conditions taken from the two-dimensional region $-1\leqslant[x,y]\leqslant1$, $v_x=v_y=0.1$, the system approaches the corresponding nth attractor. $d=-1.85$, and the total particle number is 40 000.

The exponent $h_\perp$ is computed via $h_\perp=\lim_{t\to\infty}(1/t)\ln[\delta(t)/\delta(0)]$, where $\delta(t)=\sqrt{\delta y(t)^2+[\delta v_y(t)]^2}$. In what follows we will vary d and examine the dynamics in a region where $h_\perp$ changes its sign.

Figure 1 shows the largest transverse Lyapunov exponent $h_\perp$ vs the parameter d for $d\in[-3.5,-1.0]$ in which $h_\perp$ changes from negative to positive at $d_c=-1.75$. When $d>d_c$ slightly (we choose $d=-1.70$), the system shows multistate on-off intermittency; while $d<d_c$ slightly (we choose $d=-1.85$), the system has multistate intermingled basins. $d=d_c$ is just the blowout bifurcation point in our case.

As the parameter d is a bit smaller than d_c, $h_\perp$ is slightly smaller than zero and the chaotic motions in the invariant subspaces are transversely stable; then, multistate intermingled basins can be observed. In Fig. 2, by fixing $d=-1.85$, typical time series of $y(t)$ [2(a)] and $v_y(t)$ [2(b)] from an arbitrary initial condition are plotted. Figure 2(c) shows the projection of the motion in the x-v_x plane. It shows a typical chaotic attractor of a forced-damped Duffing oscillator (5). Since d is slightly smaller than d_c, after a certain transient the typical trajectory approaches one of invariant subspaces, which is a chaotic system of Eq. (5).

Now the problem we are most interested in is to which invariant subspace the system asymptotically approaches if it starts from an arbitrary initial condition. Figure 3 shows the basins of attractors in the subspaces $y=n\pi-\pi/2(n\in\mathbb{Z})$, $v_y=0$ with different state number n (n is from -4 to 4) for the initial conditions taken in the two-dimensional region $-1\leqslant[x,y]\leqslant1$, with $v_x=v_y=0.1$. The dots in different figures represent the initial conditions from which the corresponding attractors are asymptotically approached. Apparently, the basins of attractors are intermingled, since in an arbitrarily small vicinity of the basin of any given attractor one can surely find initial conditions from which the system approaches other attractors (see Figs. 3 and 4). We have tried 40 000 initial conditions arbitrarily chosen in the region of Fig. 3 and count the number of initial conditions from which the system can reach different attractors. The distribution of these initial conditions N_n is plotted in Fig. 5(a), which follows a nice exponential decay e^{-nr} [see also Fig. 5(b)]. It shows that the particle can run very far away from the starting region. When we take the region of Fig. 4 for our counting, the exponential relation remains the same; i.e., this exponential decay law is an intrinsic and scaling invariant feature of the system.

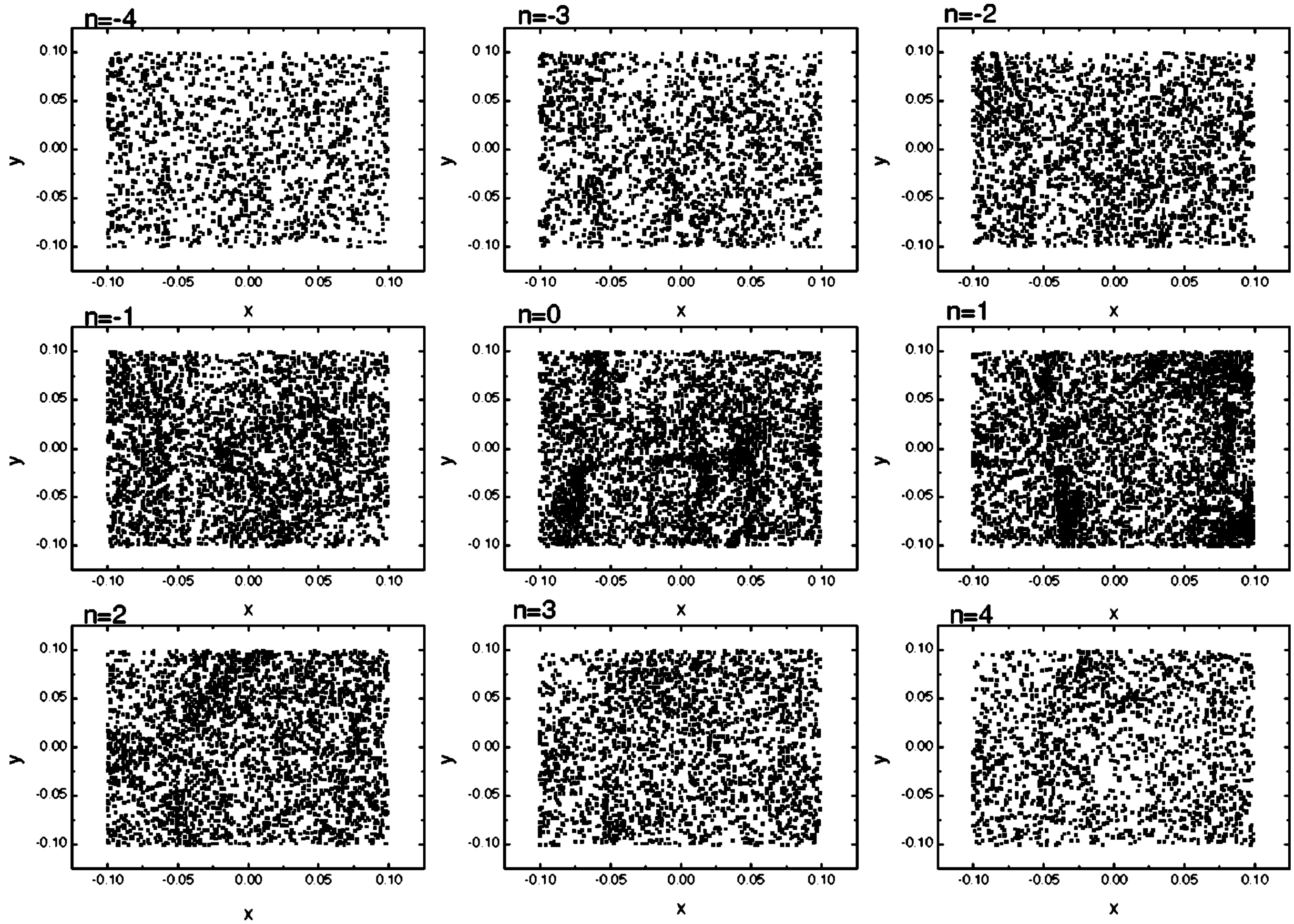

FIG. 4. The same as Fig. 3 with $-0.1 \leqslant [x,y] \leqslant 0.1$, $v_x = v_y = 0.1$ taken for the initial distribution. The riddled basin structure is further confirmed.

IV. TRANSPORT PROPERTY OF THE ON-OFF INTERMITTENCY STATE

As the parameter d increases through d_c, $h_\perp$ becomes slightly larger than zero and the motions in the invariant subspaces are no longer transversely stable. This leads to an intermittent behavior: a typical trajectory spends a long time near one of the invariant subspaces and is repelled away from this set, then can be attracted to another invariant subspace, and then repeats this on-off procedure again. In Fig. 6 by fixing $d = -1.70$, typical time series of $y(t)$ [6(a)] and $v_y(t)$ [6(b)] from an arbitrary initial condition are plotted. y

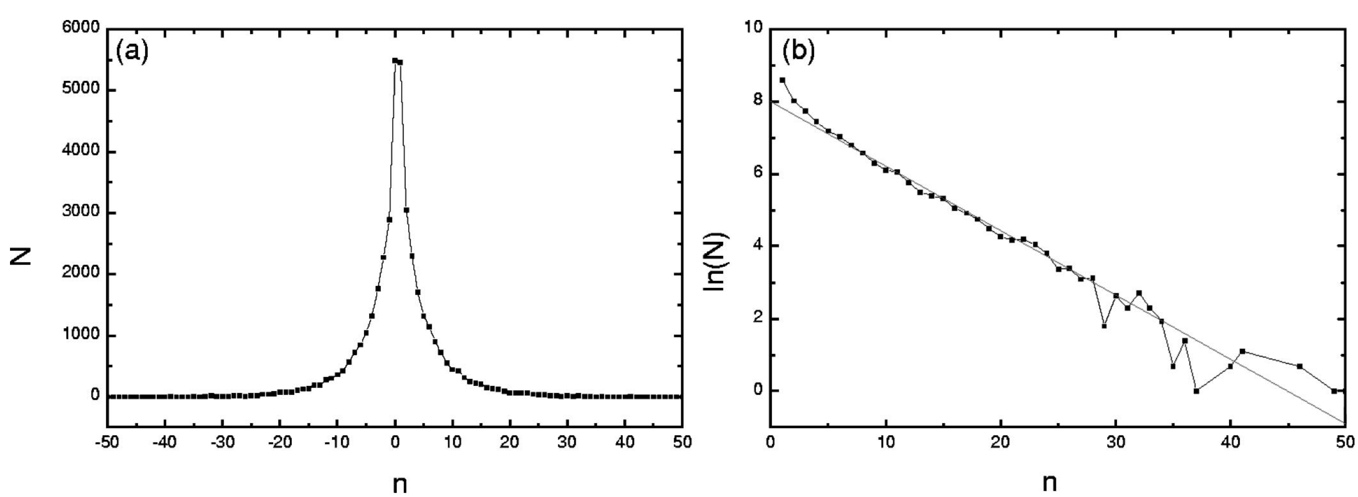

FIG. 5. (a) The particle number N_n vs the index of the invariant subspaces n. (b) The semilogarithm of $\ln(N_n)$ vs $n(50 \geqslant n \geqslant 1)$, $N_n \propto e^{-nr}$, with $r = 0.18$.

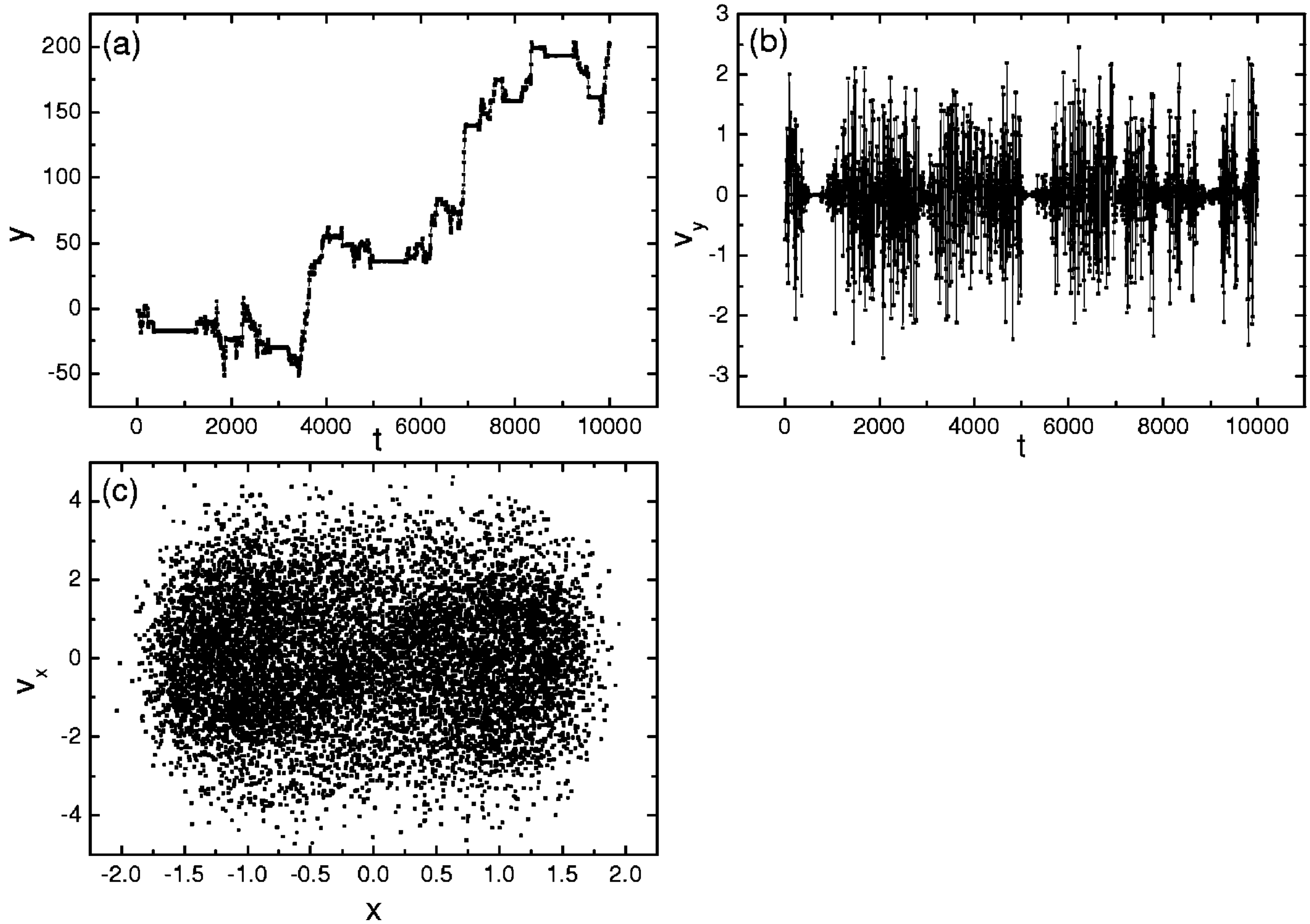

FIG. 6. (a), (b) The typical time sequencies of y and v_y vs t, respectively, with $d=-1.70$. They show obviously on-off intermittency. In particular, the steplike behavior of $y(t)$ in (a) manifests the characteristic of random walk in various states at multistate on-off intermittency. (c) The projection of the motion in the x-v_x plane, which is almost the same as Fig. 2(c).

and v_y are obviously of multistate on-off intermittency and consequently $y(t)$ has a steplike behavior which shows a typical random walk motion among different invariant sets. Figure 6(c) shows the projection picture in the x-v_x plane. It still shows the typical chaotic attractor of a forced-damped Duffing oscillator (5) the same as Fig. 2(c). Since d is slightly larger than d_c, the typical trajectory costs the most time near various invariant subspaces.

As the particle performs characteristic Brownian motion in Fig. 6(a) when multistate on-off intermittency takes place, it is interesting to study its transport property. We choose 2000 initial values with x, y, v_x, and v_y all being random numbers between -1 and 1, and plot the relation between $\ln\langle y^2(t)\rangle$ and $\ln t$. In Fig. 7 the relation obeys the following nice power law:

$$\langle y^2(t)\rangle \propto t^{2H}, \tag{7}$$

where H is the Hurst exponent which must satisfy $0<H<1$. In our case we get $H=0.46$, which is close to the exponent for ordinary Brownian motion, $H=1/2$.

Now we can compare the statistic behaviors of the system for different evolution stages in different situations: the situation exhibiting multistate intermingled basins ($d=-1.85$) and the other exhibiting multistate on-off intermittency ($d=-1.70$). We run an ensemble of systems and count how the occupations of different invariant subspaces vary as time. For small d, $d=-1.85$, Figs. 8(a), 8(b), 8(c), and 8(d) show the occupation distribution at $t=50$, 100, 1000, and 4000, respectively. The initial conditions are also

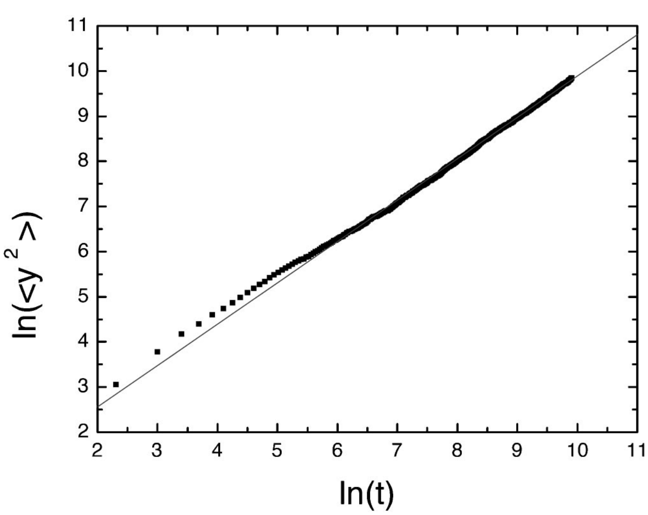

FIG. 7. $d=-1.70$, $\ln\langle y^2(t)\rangle$ vs $\ln t$. Diffusion with the Hurst exponent $H=0.46$ is observed.

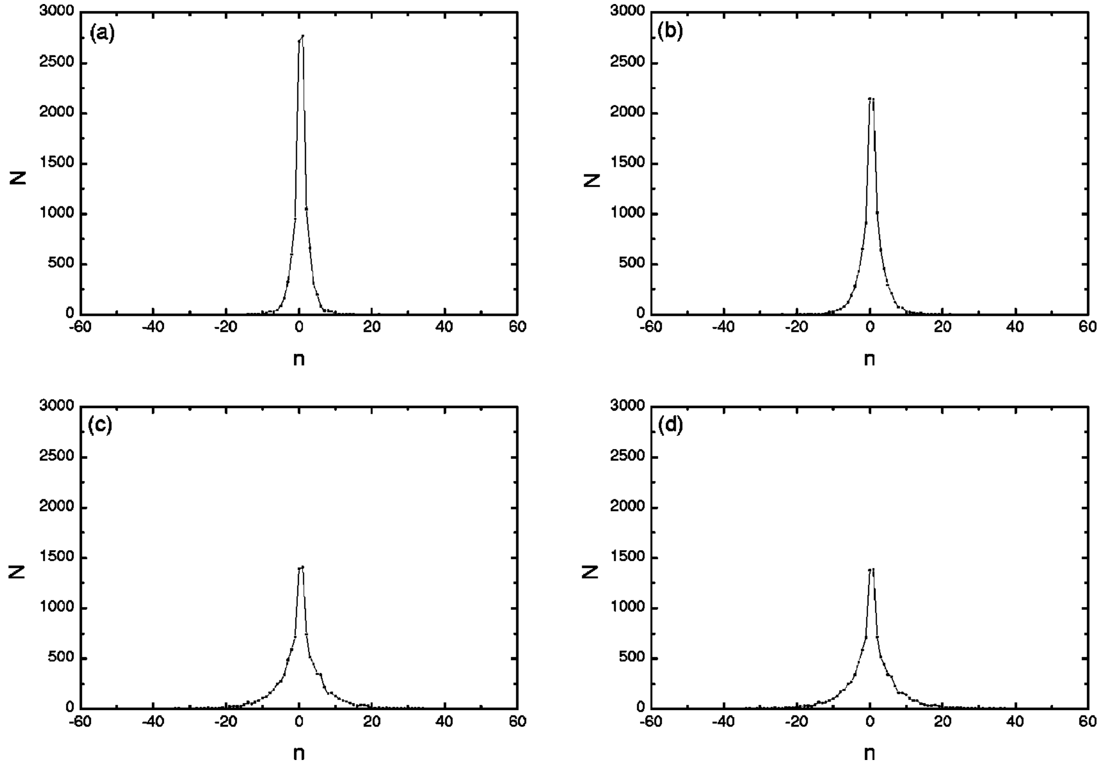

FIG. 8. (a)–(d) The dynamic statistics of N_n vs the state number n, at different times, $t=50$, 100, 1000, and 4000, respectively. $d=-1.85$, and the total particle number is 10 000.

taken from the two-dimensional region $-1 \leqslant [x,y] \leqslant 1$, with $v_x = v_y = 0.1$; the total particle number is 10 000. We identify the occupation at the nth invariant set $N_n(t)$ as the number of particles located in between $(n-1)\pi < y(t) < n\pi$. Clearly, at an early time, all of the particles concentrate at small n due to the choices of initial conditions. Then the particles start diffusing because of the intermingled basin structure. While the diffusion velocity dampens with time,

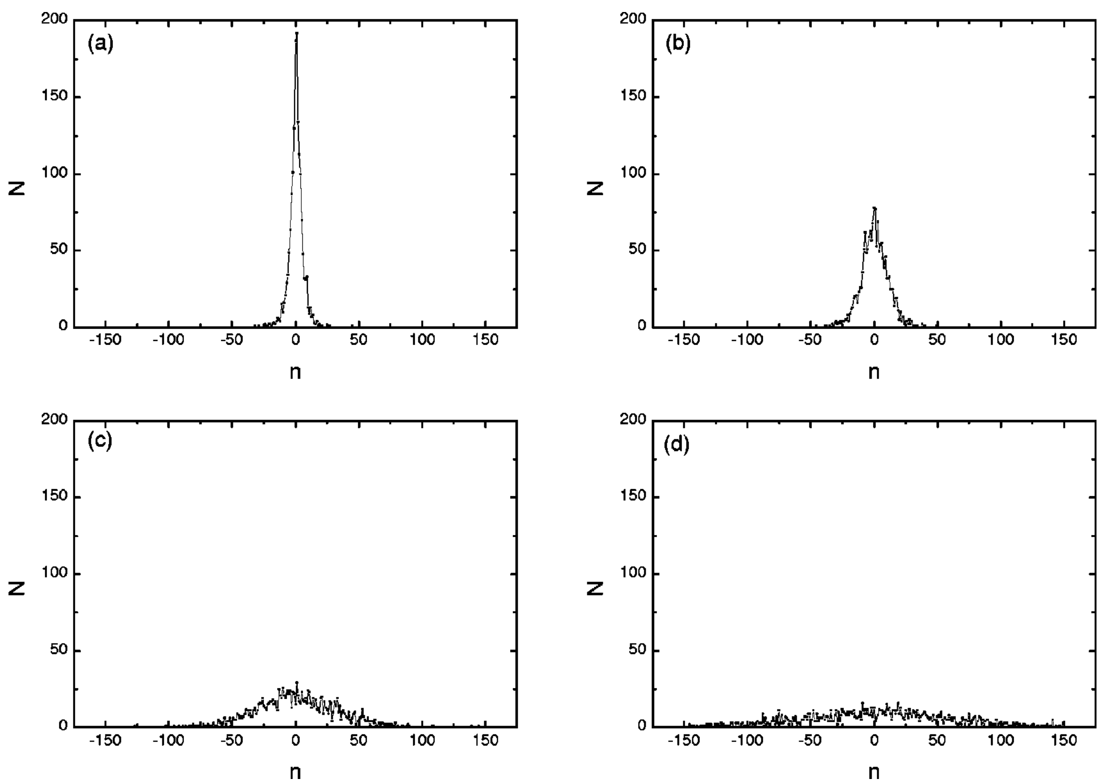

FIG. 9. (a)–(d) The same as Fig. 8, with $t=200$, 1000, 10 000, and 40 000, respectively. $d=-1.70$, and the total particle number is 1600.

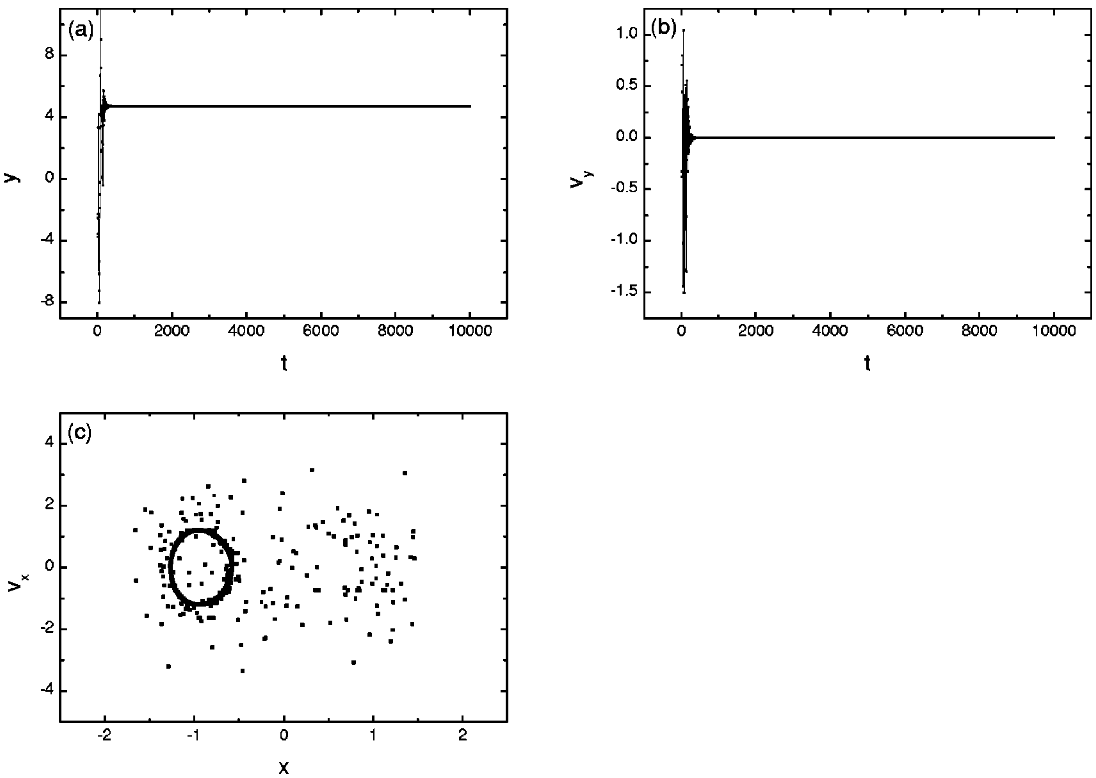

FIG. 10. (a), (b) The typical time sequencies of y and v_y vs t, respectively, with $d=-1.0$, $f_0=2.0$. $f_0=2.0$ determines that there is a limit circle in each of the invariant subspaces, while $d=-1.0$ determines that the invariant subspaces are all transversely stable. (c) The projection of the motion in the x-v_x plane.

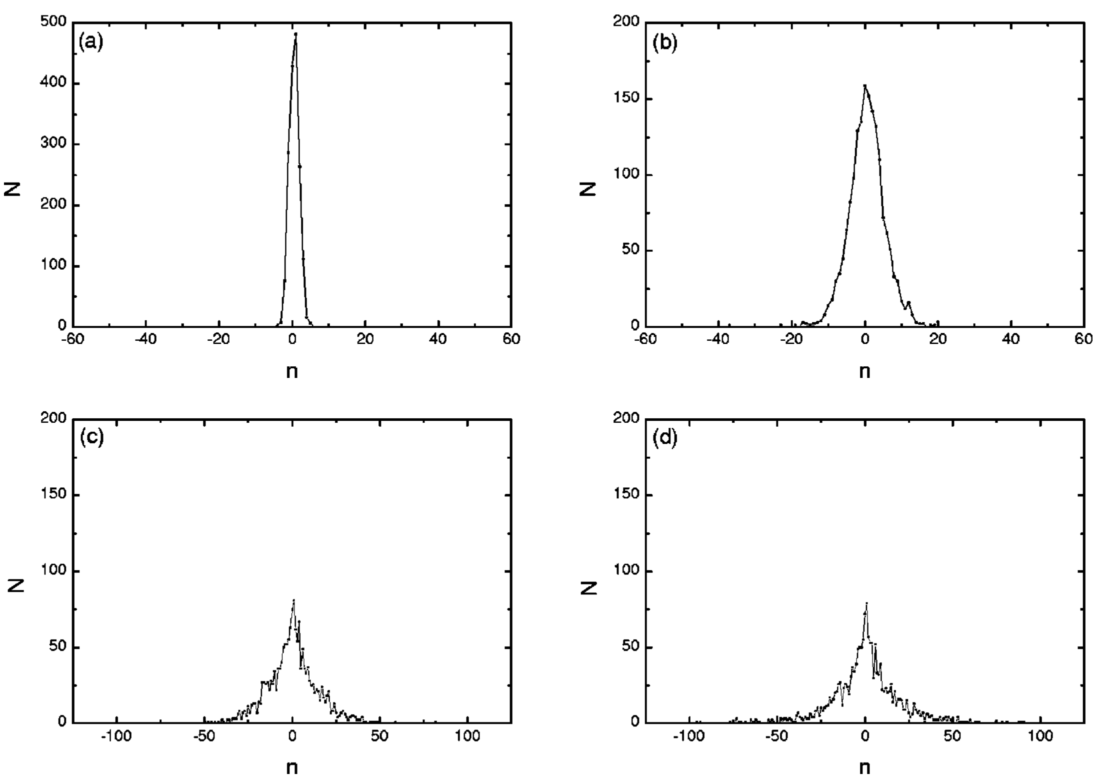

FIG. 11. (a)–(d) The same as Fig. 8, with $t=20$, 100, 1000, and 4000, respectively. $d=-1.0$, $f_0=2.0$, and the total particle number is 1600.

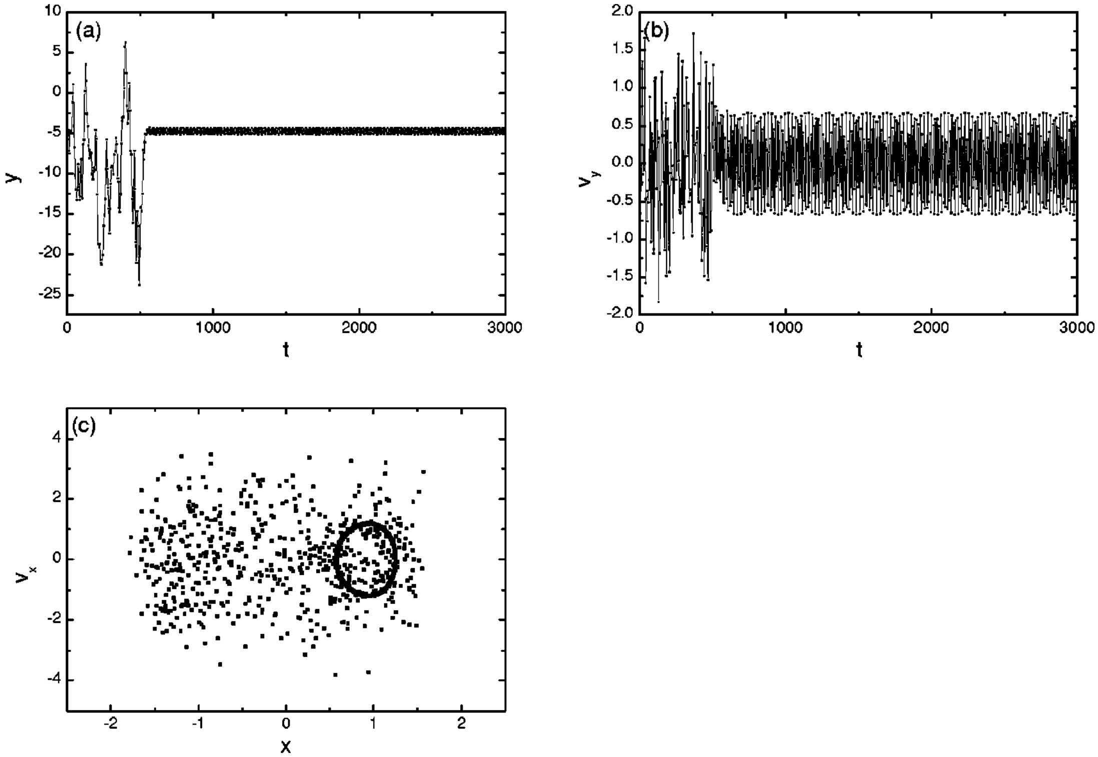

FIG. 12. (a)–(c) The same as Fig. 10, with $d=-0.6$, which shows that the transverse direction is unstable. After the transient, the particle stays near one of the invariant unstable subspaces and oscillates there forever.

for $t\rightarrow\infty$, we get a stationary statistic distribution just like Fig. 5. For $d=-1.70$, on-off intermittency becomes valid. Figures 9(a), 9(b), 9(c), and 9(d) show the occupation distributions at $t=200$, 1000, 10 000, and 40 000, respectively. Now the occupations $N_n(t)$ do not approach a peaked and localized distribution: rather, the distribution flattens as time, and for $t\rightarrow\infty$ the density of distribution becomes zero for all n and the particles can diffuse to infinity.

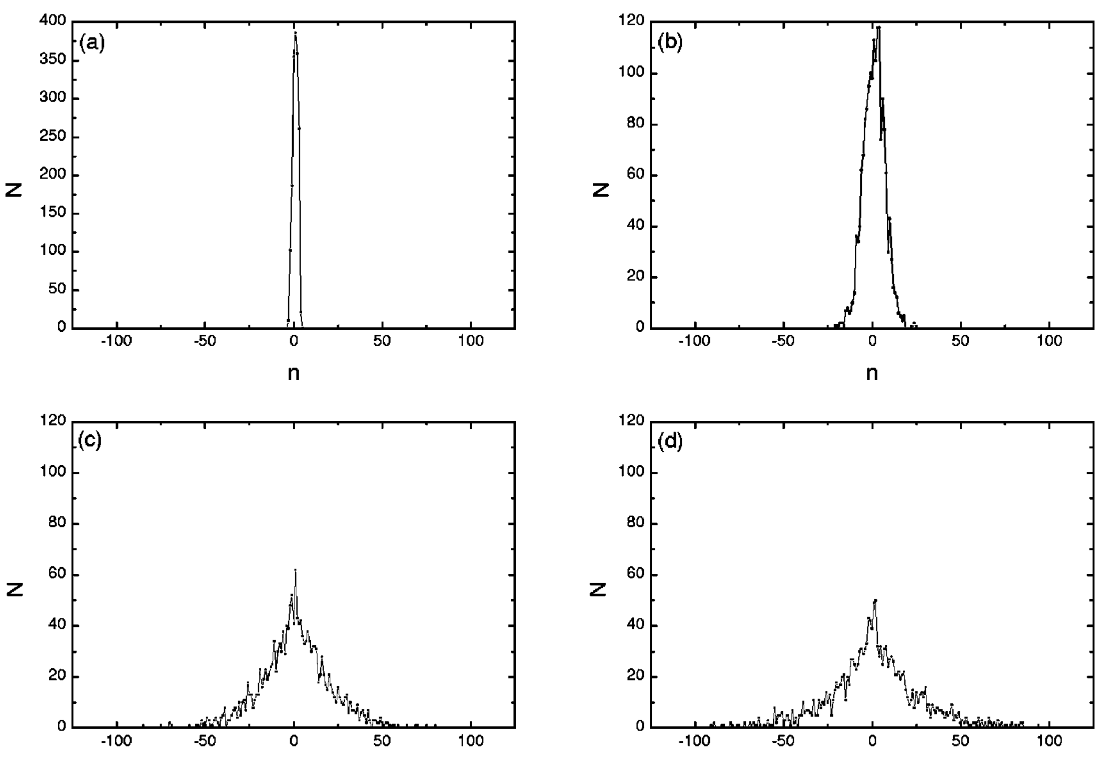

FIG. 13. (a)–(d) The same as Fig. 9, $d=-0.6$.

V. STATISTICS AND TRANSPORT FOR THE PERIODIC BASIC STATE

In all the above sections we considered cases where the motion in the invariant subspaces is chaotic [Fig. 2(c)]. It is worthwhile investigating what happens if the motion in the invariant subspace is periodic. Now we take $f_0=2.0$; then, the motion in the invariant subspaces is a limit circle. In Fig. 10, by fixing $d=-1.0$, the transverse direction is stable. We get typical time behaviors of $y(t)$ [10(a)] and $v_y(t)$ [10(b)] from an arbitrary initial condition. Figure 10(c) shows the projection of the system trajectory in the x-v_x plane, and it is a limit circle indeed. In Figs. 11(a), 11(b), 11(c), and 11(d), we get the occupation distributions at the nth invariant set $N_n(t)$ at $t=20$, 100, 1000, and 4000, respectively, in a similar way as Fig. 8. The total particle number is 1600. We find also an exponential distribution; however, we should emphasize that basins of different attracting sets are now no longer riddled by each other, but the fractal nature of the basins can be observed. For $d=-0.6$, the transverse direction becomes unstable, and typical time series of $y(t)$ and $v_y(t)$ are plotted in Figs. 12(a) and 12(b), respectively. After the transient, the particle stays near one of the invariant unstable subspaces and oscillates there. The dynamical behavior is essentially different from the multistate on-off intermittency when a strange chaotic attractor in each invariant subspace is identified (Fig. 6), which is a fractional Brownian motion in the y direction. Similar to computing the occupation distributions in Fig. 9, we plot diagrams of the periodic basic state in Fig. 13 at $t=20$, 100, 1000, and 4000, respectively. We find that the distribution approaches a finite stationary exponential distribution, and this localized distribution presents a striking contrast to the distribution in Fig. 9 which flattens to zero as time goes to infinity.

VI. CONCLUSION

In summary, we have presented a physical model which exhibits intermingled basins and on-off intermittency of multiple coexisting states when a single parameter is varied. We find that infinitely many basins of attractors can be riddled by each other when the motions on the attractors are chaotic, and the system can wander among infinite states when on-off intermittency occurs. With multistate intermingled basins we have a distribution diagram which shows clearly an exponential decay of the occupation distribution while in the case of multistate on-off intermittency the particle performs Brownian motion in the y direction. In Refs. [21] and [22], the authors studied the chaotic itinerancy phenomenon between attractor ruins. Here our approach has been to address the distribution property and transportation behavior in the presence of multi-intermingled basins and on-off intermittency.

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